SECTION 17.4: GREEN'S THEOREM

GREEN'S THEOREM (WORK/CIRCULATION FORM):

Suppose R is a simply connected region in the plane with piecewise smooth boundary C oriented counter-clockwise. Let $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ where M and N have continuous first partial derivatives. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \int_C M\,dx + N\,dy = \iint_R \left(N_x(x,y) - M_y(x,y)\right)\,dA$$

CONSTRUCTIVE DERIVATION:

NOTE: $N_x - M_y$ is called the '**scalar curl**' of $\vec{F}$ and measures the tendency for $\vec{F}$ to rotate counter-clockwise.

If $N_x - M_y = 0$, $\vec{F}$ is called '**irrotational**' since in this case, the net circulation about any curve C is:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(N_x(x,y) - M_y(x,y)\right)\,dA = \iint_R 0\,dA = 0.$$

QUESTION: Where else have we seen this?

EXAMPLE 1: Let $\vec{F}(x,y) = \langle 2y, -x \rangle$ and let C denote the circle $x^2 + y^2 = 16$.

1. Use Green's Theorem to find the counter-clockwise circulation of $\vec{F}$ along C.

Ans: $\oint_C \vec{F} \cdot d\vec{r} = \int_C 2y\,dx - x\,dy = \iint_R (-3)\,dA = -3(\text{Area of } R) = -48\pi$

2. Check your answer using the techniques from Section 17.2.

Ans: Using $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \le t \le 2\pi$, $\oint_C \vec{F} \cdot d\vec{r} = \ldots = \int_0^{2\pi} \left(-32\sin^2(t) - 16\cos^2(t)\right)\,dt = -48\pi$

3. What would be the **clockwise** circulation of $\vec{F}$ along C ?

Ans: 48π

EXAMPLE 2: Use Green's Theorem to find

$$\oint_C \left(\sin^2(3y) + 4x^2y\right)\, dy - \left(e^{3x} - xy^2\right)\, dx$$

where C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$ oriented counter-clockwise.

HINT: Rewrite the integral in the form $\displaystyle\oint_C M(x, y)\, dx + N(x, y)\, dy$ before you get too far along.

Ans: $\displaystyle\oint_C \left(\sin^2(3y) + 4x^2y\right)\, dy - \left(e^{3x} - xy^2\right)\, dx = \int_0^1 \int_0^{2x} 6xy\, dy\, dx = 3$

GREEN'S THEOREM (FLUX FORM):

Suppose R is a simply connected region in the plane with piecewise smooth boundary C oriented counter-clockwise. Let $\vec{F}(x,y) = \langle M(x,y), N(x,y) \rangle$ where M and N have continuous first partial derivatives. Then

$$\oint_C \vec{F} \cdot d\vec{n} = \int_C M\,dy - N\,dx = \iint_R \left(M_x(x,y) + N_y(x,y)\right)\,dA$$

PROOF ONE: Use Green's Theorem Circulation Form: $\displaystyle\int_C M\,dy - N\,dx = \int_C (-N)\,dx + M\,dy = \ldots$

CONSTRUCTIVE DERIVATION:

NOTE: $M_x + N_y$ is called the '**divergence**' of $\vec{F}$ and measures the tendency for $\vec{F}$ to 'spread out.'

If $M_x + N_y = 0$, $\vec{F}$ is called '**source free**,' '**solenoidal**,' or '**incompressible**.'

Under the conditions of Green's Theorem the outward flux of a source free field $\vec{F}$ across C is 0.

AREA AS A LINE INTEGRAL:

Let R be a simply connected region with boundary C, oriented counter clockwise. Use Green's Theorem to show:

$$\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx = \oint_C x\,dy = -\oint_C y\,dx$$

EXAMPLE 3: Use a line integral to find the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Ans: Using $x = a\cos(t)$ and $y = b\sin(t)$, $0 \le t \le 2\pi$, we get the area to be πab units^2.

EXAMPLE 4: Show $\dfrac{1}{2}\oint_C x\,dy - y\,dx$ is the outward flux of $\vec{F}(x,y) = \dfrac{1}{2}\langle x, y\rangle$ across C.

EXTENSIONS TO NON-SIMPLY CONNECTED REGIONS:

EXAMPLE 5: Let $\vec{F}(x,y) =< 2y, -x >$ and R be the annulus $\{(x,y) : 1 \leq x^2 + y^2 \leq 4\}$.

Let C be the boundary of R oriented counter clockwise.

1. Sketch R and C.

2. Find the counter-clockwise circulation of $\vec{F}$ along C.

$$\text{Ans: Circulation} = \oint_C M\,dx + N\,dy = \iint_R (N_x(x,y) - M_y(x,y))\,dA = \iint_R -3\,dA = -3 = \ldots = -9\pi$$

3. Find the outward flux of $\vec{F}$ across C.

$$\text{Ans: Flux} = \oint_C M\,dy - N\,dx = \iint_R (M_x(x,y) + N_y(x,y))\,dA = \iint_R 0\,dA = 0$$

STREAM FUNCTIONS / FLUX POTENTIAL

RECALL: If $\vec{F}$ is conservative, then there is a potential ϕ for $\vec{F}$ so that $\nabla\phi = \vec{F}$ and

$$\text{Work / Circulation} = \int_C \vec{F}\cdot d\vec{r} = \int_C M\,dx + N\,dy = \int_C \phi_x\,dx + \phi_y\,dy = \phi(\text{terminal point of } C) - \phi(\text{initial point of } C)$$

A **stream function**, ψ for a field $\vec{F}$ is a 'flux potential' in sense that

$$\text{Flux} = \int_C \vec{F}\cdot d\vec{n} = \int_C M\,dy - N\,dx = \psi(\text{terminal point of } C) - \psi(\text{initial point of } C)$$

If such a function ψ exists then we'd have:

$$\int_C M\,dy - N\,dx = \int_C -N\,dx + M\,dy = \int_C \psi_x\,dx + \psi_y\,dy$$

We see $\psi_x(x,y) = -N(x,y)$ and $\psi_y(x,y) = M(x,y)$. Here, $\psi_{xy}(x,y) = -N_y(x,y)$ and $\psi_{yx}(x,y) = M_x(x,y)$ so the condition $\psi_{xy}(x,y) = \psi_{yx}(x,y)$ would be equivalent to $-N_y(x,y) = M_x(x,y)$ or $M_x + N_y = 0$.

If $\vec{F} = \langle M, N\rangle$ where M and N have continuous first partials throughout a simply connected region R, we know if $N_x - M_y = 0$, that is if $\vec{F}$ is irrotational, then $\vec{F}$ is conservative so there is a function ϕ with $\nabla\phi = \vec{F} = \langle M, N\rangle$.

Under these same circumstances, if $M_x + N_y = 0$, that is, if $\vec{F}$ is source-free, then $\vec{F}$ has a stream function, ψ.

EXAMPLE 6: Show $\vec{F}(x,y) =< 2y, -x >$ is source-free and find a stream function, ψ.

Use ψ to find the flux of $\vec{F}$ across the quarter of the unit circle which lies in Quadrant I, oriented counter-clockwise.

Ans: $M_x + N_y = 0 + 0$ and $\psi(x,y) = \frac{1}{2}x^2 + y^2 + C$; Flux $= \psi(0,1) - \psi(1,0) = \frac{1}{2}$

EXAMPLE 7: Show that if $\vec{F} = \langle M, N\rangle$ where M and N have continuous first partials throughout a simply connected region R with boundary C, and ψ is a stream potential for $\vec{F}$, then the outward flux across C is 0.

HARMONIC FUNCTIONS

If $\vec{F} = \langle M, N \rangle$ is conservative with potential ϕ then $\phi_x = M$ and $\phi_y = N$.

If, in addition $\vec{F}$ has a flux potential ψ, then $M = \psi_y$ and $N = -\psi_x$. Putting these together, we get:

$$\phi_{xx} + \phi_{yy} = (\phi_x)_x + (\phi_y)_y = M_x + N_y = (\psi_y)_x + (-\psi_x)_y = \psi_{yx} - \psi_{xy} = 0$$

This shows ϕ satisfies the **Laplace (Partial Differential) Equation** and ϕ is said to be **harmonic**.

EXAMPLE 8: Show $\phi(x, y) = e^{-x} \sin(y)$ is harmonic.

Find the corresponding gradient field $\vec{F}$ and show $\vec{F}$ is both conservative and source free.

HOMEWORK: Section 17.4: 9 - 69 every other odd.